Mathematisches Institut der Universität München
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WS 19/20
Sheet 2

Exercises for Stochastic Processes

Tutorial exercises:

T1. Let M and N be independent Poisson processes with intensities λ and μ. Show that $M + N$ is a Poisson process with intensity $\lambda + \mu$.

T2. Let $X = (X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are i.i.d. standard normal random variables. Let O be an $n \times n$ orthogonal matrix. Show that $OX \overset{d}{=} X$.

T3. Consider an insurance company where claims arrive according to a Poisson process with intensity $\lambda > 0$. Suppose the size of each claim X_i is independent and identically distributed with distribution μ, with existing moment generating function

$$m_X(t) := \int_0^\infty \exp(tx)\mu(\mathrm{d}x).$$

The insurance company has a constant income rate of $c > 0$, and a starting capital of $u > 0$. Define the loading factor θ such that $c = (1 + \theta)\lambda\mathbb{E}[X_1]$, and suppose $\theta > 0$. Let X_t be the capital of the insurance company at time t. Show that the probability $\psi(u)$ that the company goes bankrupt, i.e., the probability that $X_t < 0$ for some $t > 0$, satisfies

$$\psi(u) \leq \exp(-Ru),$$

where R is the positive solution to the equation

$$1 + (1 + \theta)\mathbb{E}[X_1]R = m_X(R).$$

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Homework exercises:

H1. Let $N, X_1, X_2, \ldots$ be independent random variables, N Poisson distributed and X_k uniformly distributed on $[0,1]$. Show that

$$N_t := \sum_{k=1}^{N} 1_{[0,t]}(X_k) \qquad (t \in [0,1])$$

is a Poisson process (restricted to $t \in [0,1]$) in the sense of the "alternative 1" definition from the lecture. How can it be extended to all $t \geq 0$?

H2. (a) Let $\lambda > 0$. Consider the family of probability measures

$$\{P^{\underline{t}} \,:\, \underline{t} = (t_1, t_2, \ldots, t_n), 0 \leq t_1 \leq \cdots \leq t_n, n \in \mathbb{N}\},$$

where $P^{\underline{t}}$ is defined on $(\mathbb{N}_0^n, \mathcal{P}(\mathbb{N}_0^n))$, with the property that for all $n \in \mathbb{N}$ and for all $\underline{t}$, the random variables X_{t_1},
$X_{t_2} - X_{t_3}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent under $P^{\underline{t}}$ and have distribution

$$X_{t_1} \sim \mathrm{POI}(\lambda t_1), \quad X_{t_i} - X_{t_{i-1}} \sim \mathrm{POI}(\lambda(t_i - t_{i-1})), \quad i = 2, \ldots, n.$$

Show that the family $\{P^{\underline{t}}\}$ is compatible.

(b) Let $\sigma^2 > 0$. Consider the family of probability measures

$$\{P^{\underline{t}} \,:\, \underline{t} = (t_1, t_2, \ldots, t_n), 0 \leq t_1 \leq \cdots \leq t_n, n \in \mathbb{N}\},$$

where $P^{\underline{t}}$ is defined on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, with the property that for all $n \in \mathbb{N}$ and for all $\underline{t}$, the random variables X_{t_1},
$X_{t_2} - X_{t_3}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent under $P^{\underline{t}}$ and have distribution

$$X_{t_1} \sim \mathcal{N}(\sigma^2 t_1), \quad X_{t_i} - X_{t_{i-1}} \sim \mathcal{N}(\sigma^2(t_i - t_{i-1})), \quad i = 2, \ldots, n.$$

Show that the family $\{P^{\underline{t}}\}$ is compatible.

H3. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded measurable function and X and Y are random variables such that X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$. Show that

$$\mathbb{E}(f(X,Y) \mid \mathcal{G}) = g(X)$$

with

$$g(x) = \mathbb{E}(f(x,Y)).$$

Deadline: Monday, 28.10.19